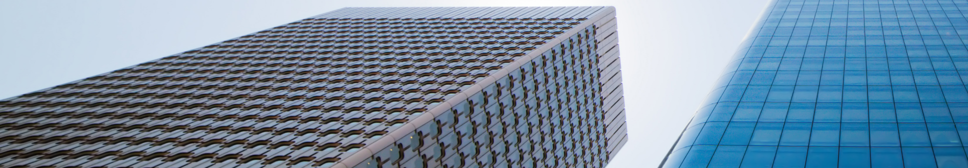
Introduction to Quantitative and Computational Finance

Module 1
Derivative products and their price

Benjamin Couéraud

Research Scientist at QuantFi



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Introduction to derivative products

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- Other examples: *interest rate derivatives* (like swaps), credit risk derivatives (like CDS).

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- In a forward contract, the two counterparties are *obliged* to deliver the underlying at the agreed price (called the *forward price*, which isn't the price of the contract itself, but the delivery price).

The payoff of a forward

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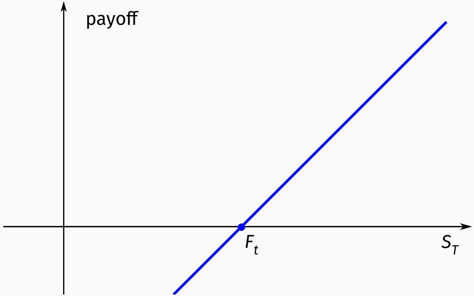
Suppose we are buying the underlying at the agreed price (so-called *long* forward contract).

$$\boxed{\text{payoff}\,(S_T) = S_T - F_t}$$



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- In both cases, I am *required* to deliver the wheat at the agreed price! Both counterparties must respect the agreement.

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- A *put option* is essentially a right to sell, whereas a *call option* is a right to buy some underlying (a *stock* generally).
- A put can be used as a protection against the decrease in value of the underlying, whereas a call can be used to bet on the increase in value of the underlying.

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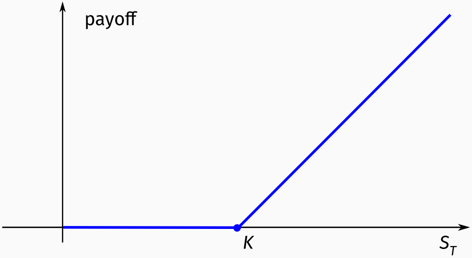
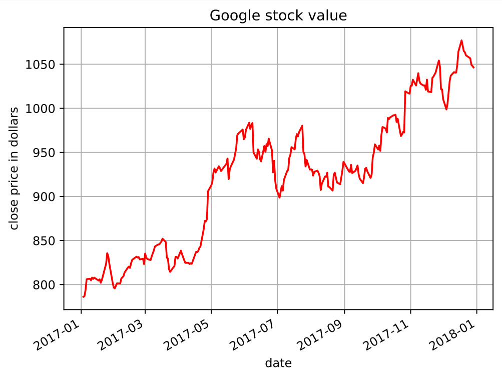


A detailed example of a call

We consider the Google stock, its value changes with time as reflected by the time series of the stock:



Observe the random behavior of the stock.

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⚠ In the case you are *selling* the call, you can potentially suffer a huge loss as the Google stock value keeps increasing (because you are giving away the benefits)!

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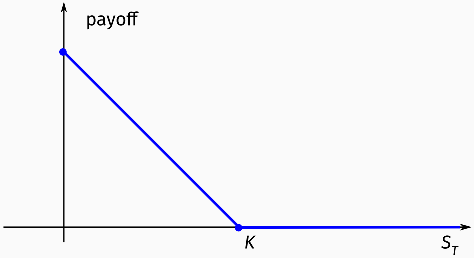

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Therefore, we limit the loss on the stock up to \$1200.

The price of a derivative product

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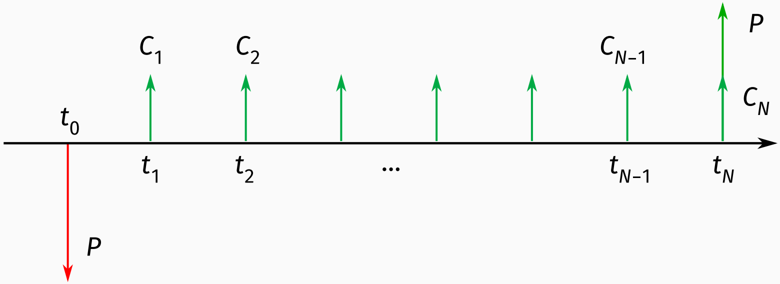
- How much should we deposit at date t, so that after T years we receive exactly the amount of money C_T? To answer we **discount** C_T to obtain its *present value*:

$$\boxed{e^{-r(T-t)}C_T} \text{ with continuous compounding.}$$

  This is the **price** we must pay at t, in order to obtain C_T at T.

Example: pricing a bond with fixed interested rate

- We want to price the following *bond*: at date t_0 we pay an amount P (called *principal*), at date t_i ($1 \leq i \leq N$) we receive a *coupon* C_i and at the last date t_N we also get the principal back.



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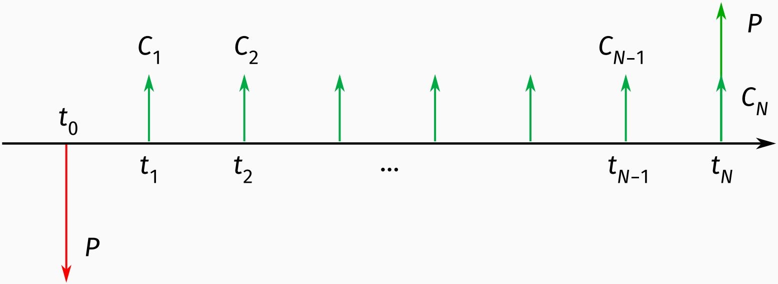

- Its price is given by discounting the series of cash flows at dates t_i, $1 \leq i \leq N$. Assuming a fixed interest rate r, the price at date t is

$$\sum_{i=1}^{N} e^{-r(t_i - t)} C_i \mathbf{1}_{t<t_i} + e^{-r(t_N - t)} P \mathbf{1}_{t<t_N}.$$

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- Risk-neutral pricing is a framework that takes into account the randomness associated with the cash flows and proposes a mathematical meaning to computing the price of a claim.

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- $\mathbb{Q}$ is engineered so as to match the market prices, which are considered as fair. Under certain conditions, $\mathbb{Q}$ is *unique*, so we can use it to price any other claim.

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- But for $C_t \equiv S_t$ a stock, we obtain that

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- We change the probability $\mathbb{P}$ to some $\mathbb{Q}$, so that today's price of the stock $(e^{-rt}S_t)_{t\geq 0}$ is a $\mathbb{Q}$-martingale: the stock trend disappears under this point of view, making trading on the stock a fair game.

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- The price of C_T at date t could be $\mathbb{E}\left[e^{-r(T-t)}C_T \,\middle|\, \mathcal{F}_t\right]$ where $(\mathcal{F}_t)_{t\geq 0}$ represents all available information up to time t. It's a random variable because we still don't know what will happen between t and T.
- But for $C_t \equiv S_t$ a stock, we obtain that

$$\mathbb{E}\left[e^{-rT}S_T \,\middle|\, \mathcal{F}_t\right] = e^{-rt}S_t \quad \forall T > t,$$

  meaning the discounted stock value $(e^{-rt}S_t)_{t\geq 0}$ is a $\mathbb{P}$-**martingale**, which isn't the case in practice: betting on the stock market isn't a fair game.
- We change the probability $\mathbb{P}$ to some $\mathbb{Q}$, so that today's price of the stock $(e^{-rt}S_t)_{t\geq 0}$ is a $\mathbb{Q}$-martingale: the stock trend disappears under this point of view, making trading on the stock a fair game.
- So the price at date t is given by $\mathbb{E}^{\star}\left[e^{-r(T-t)}C_T \,\middle|\, \mathcal{F}_t\right]$.

The fundamental theorem of asset pricing

Arbitrage in a market occurs when one can make a gain with no initial investment. A market is **complete** when any payoff can be replicated with a particular set of liquid, exchange-traded assets.

Theorem

In an arbitrage-free market, there exists a *risk-neutral measure* $\mathbb{Q}$, equivalent to $\mathbb{P}$, such that $(e^{-rt}S_t)_{t\geq 0}$ is a $\mathbb{Q}$-martingale. If in addition the market is complete, then $\mathbb{Q}$ is unique. In that case, the *price process* $(C_t)_{t\geq 0}$ of any derivative with payoff C_T on the stock is defined by the *risk-neutral pricing formula*

$$\boxed{C_t = \mathbb{E}^\star\left[e^{-r(T-t)}C_T \,\middle|\, \mathcal{F}_t\right]}$$

where $\mathbb{E}^\star$ denotes expectation relatively to $\mathbb{Q}$.

Consequence: the discounted price process is a $\mathbb{Q}$-martingale.

Illustration: pricing a forward on a stock

- The forward price F_t is determined so that the price of the forward contract at t is 0, meaning that it is fair for all counterparties:

$$0 = \mathbb{E}^{\star}\left[e^{-r(T-t)}(S_T - F_t) \,\middle|\, \mathcal{F}_t\right]$$

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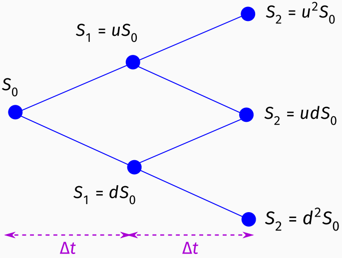
Pricing options
with the binomial model

The binomial model

- This is a *discrete* model (time and stock value) for pricing options. Over Δt the stock can goes up at increasing rate u with probability p and goes down at decreasing rate d with probability $1 - p$.



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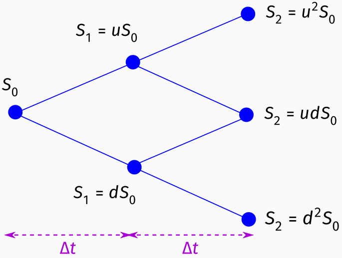

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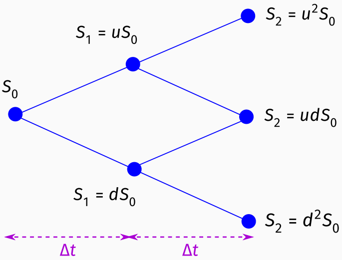

- No arbitrage is equivalent to $u > e^{r\Delta t} > d$.
- Let $(q, 1 - q)$ be the risk-neutral probability $\mathbb{Q}$. Since $(e^{-rt}S_t)_{t\geq 0}$ is a $\mathbb{Q}$-martingale, over the first period we have:

$$S_0 = \mathbb{E}^\star[e^{-r\Delta t}S_1] = e^{-r\Delta t}\big(quS_0 + (1 - q)dS_0\big) \Rightarrow q = \frac{e^{r\Delta t} - d}{u - d}$$

Pricing European path-independent options

- ☑ $(S_t)_{t\geq 0}$ is a Markov process and $C_T = \Lambda(S_T)$ so we can write the *price process* as $C_t = P(t, S_t)$ where P is the *price function*:

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- Using this formula we can price the derivative *backwards*, starting from the leaves on the right of the tree (at time T) and going back to the root node on the left (at $t = 0$).

A detailed example for a European call

$$S_0 = \$100$$

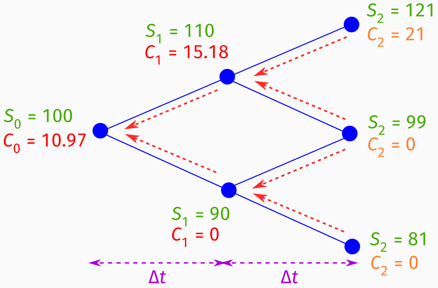
$$u = 1.1,\ d = 0.9$$

$$K = \$100$$

$$r = 0.05$$

$$\Delta t = 1$$

$$q = 0.76$$



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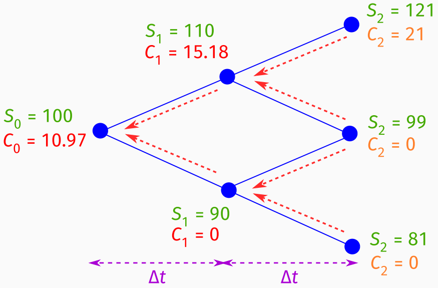

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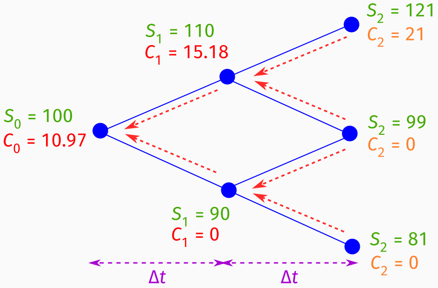

1. Compute all the possible stock values.
2. Compute all the possible payoffs: for a call $\Lambda(x) = \max(x - K, 0)$.
3. Starting from the right, apply the martingale formula to get the possible prices at previous nodes till you reach the desired time:

$$P(1,110) = e^{-r\Delta t}\big(q\Lambda(121) + (1-q)\Lambda(99)\big)$$
$$P(1,90) = e^{-r\Delta t}\big(q\Lambda(99) + (1-q)\Lambda(81)\big)$$
$$P(0,100) = e^{-r\Delta t}\big(qP(1,110) + (1-q)P(1,90)\big)$$

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- To price American options we use the same algorithm with a twist:

$$C_t = \max\left\{\Lambda(S_t),\ \mathbb{E}^\star\left[e^{-r\Delta t} C_{t+\Delta t} \,\middle|\, \mathcal{F}_t\right]\right\}$$

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- We can require the matching of the first two $\mathbb{P}$-moments of log-returns $Y_j = \log \frac{S_j}{S_{j-1}}$, assuming they are written as $\mu \Delta t$ and $\sigma^2 \Delta t$, resp., with μ the mean and σ^2 the variance of the observed log-returns.

$$\begin{cases} \mathbb{E}[Y_j] = p \log u + (1-p) \log d \equiv \mu \Delta t \\ \mathrm{Var}(Y_j) = p(1-p)(\log u - \log d)^2 \equiv \sigma^2 \Delta t \end{cases}$$

For $p = \frac{1}{2}$ we solve this system in $\log u$, $\log d$ to obtain:

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- The mean μ and standard deviation σ are computed from the log-returns time series using *statistical estimators*.