# Introduction to Quantitative and Computational Finance

*Exercises for module 1 – Derivative products and their price*

### 📝 Payoff of a bull spread – Multiple choice question (1 point)

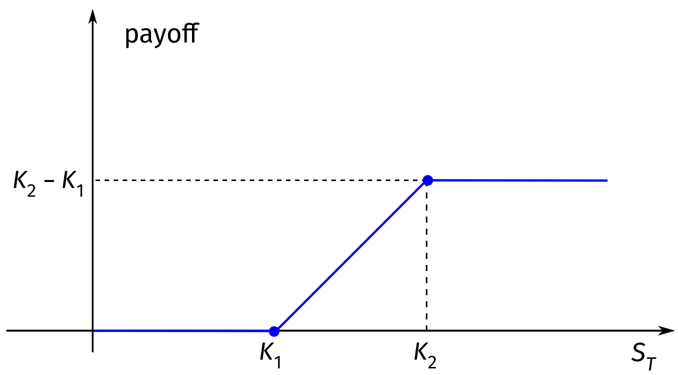
You buy a call with a strike $K_1$ and some maturity. You sell a second call with a strike $K_2 > K_1$ and the same maturity. What is the payoff of this strategy?

1. $(S_T - K_2)\mathbf{1}_{S_T \geq K_2} - \mathbf{1}_{S_T < K_1}$
2. $1 - \mathbf{1}_{S_T < K_1} + (S_T - K_1)\mathbf{1}_{K_1 \leq S_T < K_2} + (K_2 - K_1)\mathbf{1}_{S_T \geq K_2}$
3. $-\mathbf{1}_{S_T \geq K_2} + (S_T + K_2 - K_1)\mathbf{1}_{K_1 \leq S_T < K_2}$

**Answer:**

The payoff of the strategy is $C_T = \max(S_T - K_1, 0) - \max(S_T - K_2, 0)$. Therefore, we need to understand what happens when the underlying is lower than $K_1$, between $K_1$ and $K_2$ and higher than $K_2$:

- When $S_T < K_1$, $\max(S_T - K_1, 0) = 0$. But since $K_1 < K_2$, we also have $S_T < K_2$ and so $\max(S_T - K_2, 0) = 0$. Therefore, $C_T = 0$ in this case.
- When $K_1 \leq S_T < K_2$, $\max(S_T - K_1, 0) = S_T - K_1$ but $\max(S_T - K_2, 0) = 0$. Therefore, $C_T = S_T - K_1$ in this case.
- When $S_T \geq K_2$, $S_T > K_1$ too, so $C_T = (S_T - K_1) - (S_T - K_2) = K_2 - K_1$.




You will need the next result to tackle the following two exercises.

**The law of one price**

In a arbitrage-free market, one can show the following theorem, known as the *the law of one price*. Consider two assets $X$ and $Y$, as stochastic processes on $[0, T]$. Then if the values at date $T$ of these two assets are the same in all scenarios, meaning $X_T(\omega) = Y_T(\omega)$ for all $\omega \in \Omega$ where $\Omega$ denotes the universe of possible events (scenarios), then they have the same price at date $t = 0$, meaning $\text{price}_0(X) = \text{price}_0(Y)$ (which are real numbers). The result remains true if one replaces $=$ by $\leq$ or $\geq$.

### 📝 A not so exotic derivative – Two multiple choice questions (2 points)

Consider the equity derivative with payoff given by:

$$C_T = \min(\max(S_T, K_1), K_2),$$

with $K_1 < K_2$. What is the payoff of this product and in terms of payoffs of calls and/or puts?

1. $K_1 + \max(S_T - K_1, 0) - \max(S_T - K_2, 0)$
2. $K_2 - K_1 + \max(S_T - K_2 - K_1, 0)$
3. $\max(S_T - K_1, 0) - \max(K_2 - S_T, 0)$

Assuming as always that the market is arbitrage-free and complete, what is its price at $t < T$? *Hint:* use the law of one price.

1. $K_1 e^{-r(T-t)} + \text{Call}_t(T, K_1) - \text{Call}_t(T, K_2)$
2. $(K_2 - K_1)e^{-r(T-t)} + \text{Call}_t(T, K_1 + K_2)$
3. $\text{Call}_t(T, K_1) - \text{Put}_t(T, K_2)$

where $\text{Call}_t(T, K)$ denotes the price of the European put of maturity $T$ and strike $K$ at time $t$, and $\text{Put}_t(T, K)$ denotes the price of the European put of maturity $T$ and strike $K$ at time $t$.

**Answer:**

We proceed as in the previous exercise, by going through all possible cases:

- When $S_T < K_1$, then $\max(S_T, K_1) = K_1$ and $C_T = \min(K_1, K_2) = K_1$.
- When $K_1 \leq S_T < K_2$ we have $C_T = \min(S_T, K_2) = S_T$.
- When $S_T \geq K_2$, we obtain $C_T = \min(S_T, K_2) = K_2$.

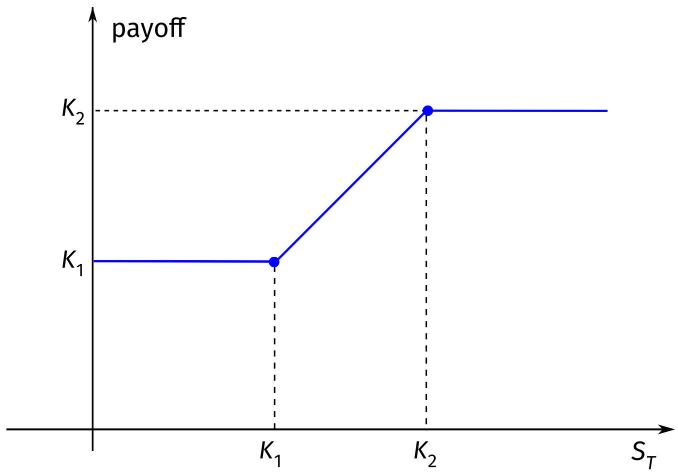
This payoff can also be written as:

$$K_1 + \max(S_T - K_1, 0) - \max(S_T - K_2, 0),$$

which is the payoff of a bull spread shifted by $K_1$, so the no-arbitrage price of this derivative at $t$ is:

$$K_1 e^{-r(T-t)} + \text{Call}_t(T, K_1) - \text{Call}_t(T, K_2),$$

where $\text{Call}_t(T, K)$ denotes the price at $t$ of a call of maturity $T$ and strike $K$.




### 📝 **Put-Call parity – Multiple choice question (1 point)**

Suppose the market is arbitrage-free and complete. Which one of the following equalities is true?

1. $\text{Call}_t(T, K) + \text{Put}_t(T, K) = S_t - K$
2. $\text{Call}_t(T, K) - \text{Put}_t(T, K) = S_t - Ke^{-r(T-t)}$
3. $\text{Put}_t(T, K) - \text{Call}_t(T, K) = e^{-r(T-t)} S_t$

where $\mathrm{Call}_t(T, K)$ denotes the price of the European put of maturity $T$ and strike $K$ at time $t$, and $\mathrm{Put}_t(T, K)$ denotes the price of the European put of maturity $T$ and strike $K$ at time $t$.

**Answer:**

We have the identity $\max(S_t - K, 0) - \max(K - S_T, 0) = S_T - K$. Therefore, using the law of one price we obtain the desired result. Note that the price at $t$ of $S_T$ is $S_t$ since $(e^{-rt}S_t)_{t\geq 0}$ is a $\mathbb{Q}$-martingale by definition of $\mathbb{Q}$.

**A general relation for European puts – Multiple choice question (1 point)**

Suppose the market is arbitrage-free and complete. Which one of the following inequalities is true?

1. $\max(Ke^{-r(T-t)} - S_t, 0) \leq \mathrm{Put}_t(T, K) \leq Ke^{-r(T-t)}$
2. $\max(K - S_t, 0) \leq \mathrm{Put}_t(T, K) \leq K$
3. $\max(Ke^{-r(T-t)} - S_t, 0) \geq \mathrm{Put}_t(T, K) \geq Ke^{-r(T-t)}$

where $\mathrm{Put}_t(T, K)$ denotes the price of the European put of maturity $T$ and strike $K$ at time $t$. Hint: use the law of one price.

**Answer:**

At maturity we have:

$$K - S_T \leq \max(K - S_T, 0) \leq K.$$

Using the absence of arbitrage, we can price these payoffs at date $t$ to obtain the inequality:

$$Ke^{-r(T-t)} - S_t \leq \mathrm{Put}_t(T, K) \leq Ke^{-r(T-t)},$$

and since $\mathrm{Put}_t(T, K) \geq 0$, we obtain the desired inequality.

**A forward contract with discrete dividends – Multiple choice question (1 point)**

We consider a long forward contract with maturity $T$ on a stock that is paying dividends $D_i$ at dates $t_i$ such that $0 < t_1 < t_2 < \cdots < t_n \leq T$. What is the forward price?

1. $F_t = e^{r(T-t)}S_t - \sum_{i=1}^{n} D_i e^{r(T-t_i)}$
2. $F_t = e^{r(T-t)}S_t + \sum_{i=1}^{n} D_i$
3. $F_t = e^{r(T-t)}S_t - \sum_{i=1}^{n} D_i$

**Answer:**

Dividends are capitalized during the life of the contract, but we can't benefit from them, since we will only hold the stock at maturity. Therefore, they need to be substracted in the payoff, which at maturity is given by:

$$S_T - \sum_{i=1}^{n} D_i e^{r(T-t_i)} - F_t.$$

Then following the same reasoning as in the course, we obtain that:

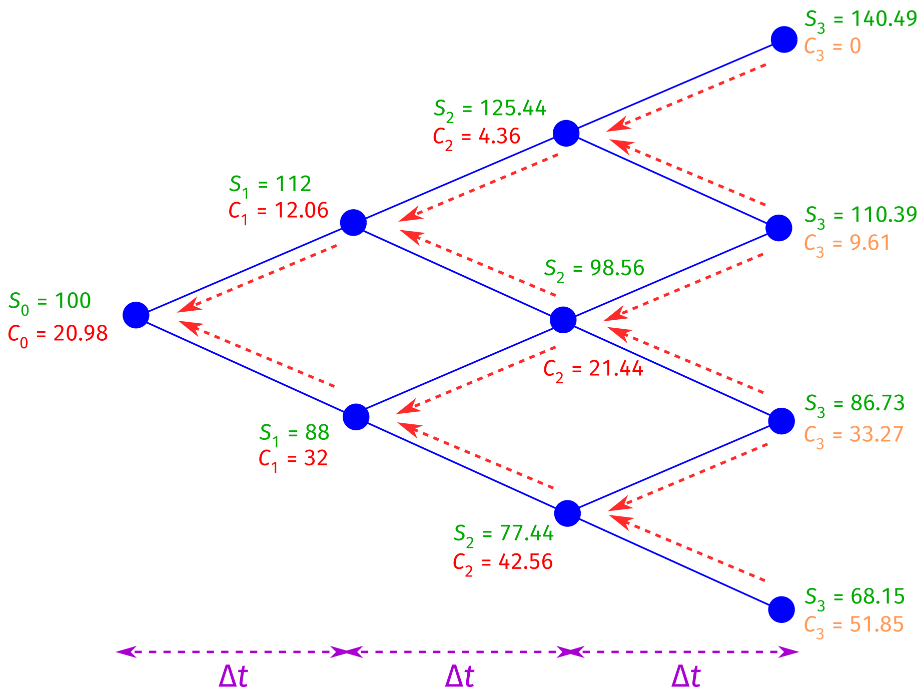
$$F_t = e^{r(T-t)}S_t - \sum_{i=1}^{n} D_i e^{r(T-t_i)}.$$

**Pricing an american option in the binomial model – Free choice question (2 points)**

We consider an American put on some stock, of strike $K = €120$ and maturity $T = 3\Delta t$, with $\Delta t = 1$ month. Price this derivative (with two decimals) using a 3-period binomial model with parameters $u = 1.12$, $d = 0.88$, $r = 0.01$ and when the spot price of the stock is $S_0 = €100$.

**Answer:**

First we compute the price of possible values for the stock. Then we compute the payoffs at the leaves of




the tree. The risk-neutral probability is $q = 0.54$. Finally, using the backward equation for American options given in the course, we obtain that the price is €20.98 (accept answer if between 20.90 and 21.00), at the root of the tree ($t = 0$).